)UESTION 1

a) Find (i) $\int \frac{1}{\sqrt{x^2+9}}\,dx$ 1

(ii) $\int \cos^3 x\,dx$ 2

(iii) $\int \frac{2x+5}{x^2+4x+13}\,dx$ 3

(b) Use the 't' method to evaluate $\int_0^{\frac{\pi}{2}} \frac{4}{3+5\cos x}\,dx$ 5

(c) (i) Given that $I_n = \int_0^3 \frac{x^n}{\sqrt{4-x}}\,dx$,

show that $I_n = \frac{2}{2n+1}(4nI_{n-1} - 3^n)$. 2

(ii) Use your result to evaluate $\int_0^3 \frac{x^2}{\sqrt{4-x}}\,dx$, 2

QUESTION 2 Start a new booklet

(a) (i) Shade on an Argand diagram the region for which 2

$|Z| \le 2$ and $1 \le \operatorname{Im} Z \le 2$.

(ii) Write down in the form $a + ib$ the complex number with least argument that lies in this region. 1

(b) You are given a complex number $Z = (1+\sqrt{3}i)(1+i)$

(i) Express Z in the form $x + iy$ where x and y are real. 1

(ii) Express $1+\sqrt{3}i$ and $1+i$ in modulus-argument form first, 2

then show that $Z = \sqrt{8}\ cis\frac{7\pi}{12}$

(iii) Hence find the exact values of $\cos\frac{7\pi}{12}$ and $\sin\frac{7\pi}{12}$ 2

(iv) What is the smallest positive integral n for which Z^n is real? 1

(c) Given two complex numbers Z_1 and Z_2 such that $Z_1 = kiZ_2$, (k is real) 2

explain why $|Z_1 - Z_2| = |Z_1 + Z_2|$

(d) (i) Find the 2 square roots of $5-12i$ and express your answers in the form $a+ib$ 2

(ii) Show the points P and Q representing these square roots on an Argand diagram. 2

Find the complex numbers represented by points R and S such that triangles PQR and PQS are equilateral.

QUESTION 3 **Start a new booklet**

(a) On a hyperbola the distance between its vertices is 6 units and the distance between its foci is 10 units.

Find (i) the distance between its directrices **2**

(ii) the acute angle between its asymptotes **2**

(b) Points P and Q are the endpoints of a focal chord of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

If the parameters at points P and Q are θ and φ, show that the eccentricity e is given by $\dfrac{\sin(\theta-\varphi)}{\sin\theta-\sin\varphi}$ **3**

(c) The equation $x^3+2x-1=0$ has roots α, β and γ.

Write down an equation with roots

(i) $-\alpha, -\beta, -\gamma$ **1**

(ii) $\alpha, -\alpha, \beta, -\beta, \gamma$ and $-\gamma$ **1**

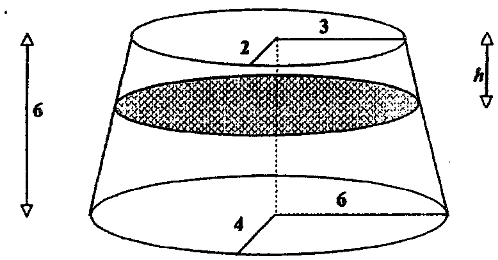
(iii) Use your answer to (ii) to find the value of $\Sigma\alpha\beta$, that is the sum of the products of the 6 roots taken 2 at a time. **2**

(d) A certain right solid has elliptical cross section as shown in the diagram. **4**

The base has semi-major axis 6 units and semi-minor axis 4 units, while the upper face has semi-major axis 3 units and semi-minor axis 2 units.

The height of the solid is 6 units.

Find an expression for the area of a typical cross section h units below the upper face, and hence by integration show that the volume of the solid is 84π cub.units



QUESTION 4 **Start a new booklet**

(a) It is given that $3-i$ is a zero of the function $P(x)=x^3+ax^2+bx-10$ where a and b are real. **3**

Find the other zeros and the values of a and b.

(b) (i) If $P(x)=x^3-9x^2+24x+c$ for some real c, find the values of x for which $P'(x)=0$. Hence find the two values of c for which $P(x)=0$ has a repeated root. **2**

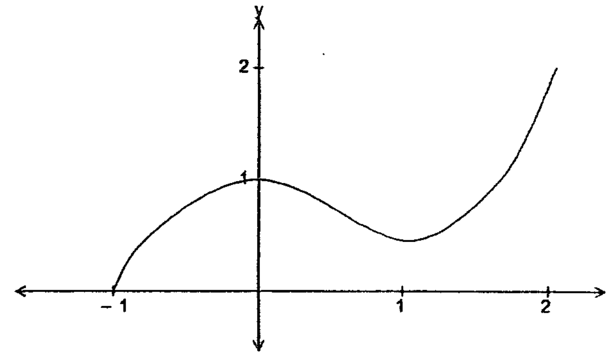
(ii) Sketch the graphs of $y=P(x)$ for these values of c. Hence write down the values of c for which $P(x)=0$ has 3 distinct real roots. **3**

(c) (i) It is given that a curve has equation $y^2=x^2(x+3)^3$. Draw the graph of the curve, paying particular attention to the shape of the curve as it crosses the axes. (Show any turning points, but you are not required to show testing their nature.) **4**

(ii) Calculate the area completely enclosed between the curve and the x axis. **3**

QUESTION 5 **Start a new booklet**

(a) The function $y=f(x)$ is defined for $-1\le x\le 2$ as shown below.



Copy the diagram onto your paper, and use the same scale to draw the graphs of

(i) $y=\tan^{-1}f(x)$ **1**

(ii) $y+1=2f(x-2)$ **2**

(iii) (ii) $2y=f(2x)$ **2**

(b) (i) Prove that $\tan^{-1}(n+1)-\tan^{-1}n=\cot^{-1}(1+n+n^2)$ **1**

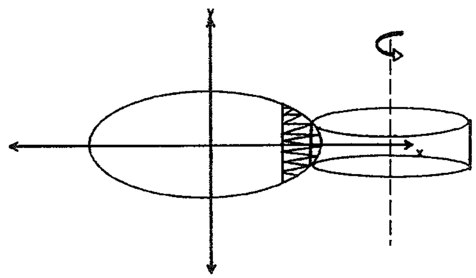
(ii) Hence show that the sum of the infinite series **2**

$$\cot^{-1}3+\cot^{-1}7+\cot^{-1}13+\dots\dots\dots+\cot^{-1}(1+n+n^2)=\cot^{-1}\left(1+\frac{2}{n}\right)$$

(c) An ellipse has equation $\dfrac{x^2}{9}+\dfrac{y^2}{5}=1$

(i) Calculate its eccentricity, and write down the equation of a directrix and the coordinates of the corresponding focus. 2

(ii) The smaller segment cut off the ellipse by its latus rectum is rotated about the corresponding directrix. 5
Using the method of cylindrical shells, find the volume of the solid generated.



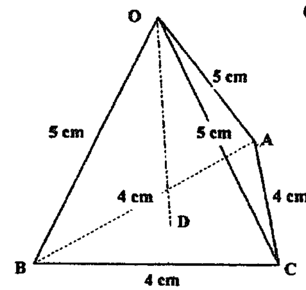
QUESTION 6 Start a new booklet

(a) (i) Given that $z=\cos\theta+i\sin\theta$, simplify the expression z^n+z^{-n} 1

(ii) Hence or otherwise solve $5z^4-11z^3+16z^2-11z+5=0$ giving your answers in the form $a+ib$. 4

(b) For this question, you may use the following information:
The median of a triangle is an interval joining a vertex to the midpoint of the opposite side. The medians intersect at a point called the **centroid**, which divides each median in the ratio 2:1.

The base of a tetrahedron OABC is an equilateral triangle of side 4 cm.
The other edges are all 5 cm.
The altitude is OD, where D is the centroid of the base ABC.



Calculate

(i) The angle OA makes with base ABC. 2

(ii) The shortest distance between the edges OA and BC. 2

(c) The point $P(x_1,y_1)$ is one point of intersection of the rectangular hyperbolas $x^2-y^2=2$ and $xy=1$.
The tangent at P to the first hyperbola cuts its asymptotes at A and C, and the tangent at P to the second hyperbola cuts its asymptotes at B and D.

(i) Draw a sketch illustrating the given information and show that the equation of AC is $xx_1-yy_1=2$ 1

(ii) Show that the coordinates of B and D are $(0,2y_1)$ and $(2x_1,0)$ 2

(iii) Prove that A, B, C and D are the vertices of a square. 3

QUESTION 7 Start a new booklet

(a) An object of mass 20 kg is dropped through the atmosphere, meeting an air resistance of $2v$ Newtons at speed v m/sec.
Acceleration due to gravity is 10m/sec^2.

(i) What is the terminal velocity? 1

(ii) Find an expression for velocity v at time t seconds. 3

(iii) Show that the distance x m travelled with velocity v is given by

$$x = 1000\log\left(\frac{100}{100-v}\right) - 10v$$ 3

(iv) Use parts (ii) and (iii) to calculate the distance the object falls in the first 10 seconds. 2

(b) The n$^{\text{th}}$ derivative of a function $f(x)$ is $\frac{d^n}{dx^n} f(x) = \frac{d^{n-1}}{dx^{n-1}}\left[\frac{d}{dx} f(x)\right]$ etc.

(i) Show that $\frac{d^n}{dx^n}(x^n) = n!$ 1

(ii) Prove by induction that for positive integral n 5

$$\frac{d^n}{dx^n}\left(x^n \ln x\right) = n!(\ln x + 1 + \frac{1}{2} + \dots\dots\dots + \frac{1}{n})$$

PYMBLE LC.
HMX2 2002
TRIAL SOLUTIONS.

Q1. (a)

(i) $\int \frac{1}{\sqrt{x^2+9}}\,dx = \ln\left(x+\sqrt{x^2+9}\right) + C$ (1)

(ii) $\int \cos^3 x\,dx = \int (1-\sin^2 x)\cos x\,dx$

$= \sin x - \frac{1}{3}\sin^3 x + C$ (1) (1)

or $\int \cos^3 x\,dx = \int \cos^2 x \cdot \frac{d}{dx}(\sin x)\,dx$

$= \sin x \cos^2 x + \int 2\sin^2 x \cos x\,dx$

$= \sin x \cos^2 x + \frac{2}{3}\sin^3 x + C$ (1) (1)

(iii) $\int \frac{2x+5}{x^2+4x+13}\,dx = \int \frac{(2x+4)+1}{(x^2+4x+4)+9}\,dx$ (1)

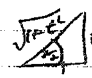
$= \ln(x^2+4x+13) + \frac{1}{3}\tan^{-1}\left(\frac{x+2}{3}\right) + C$ (1) (1)

$\int_0^{\frac{\pi}{2}} \frac{4}{3+5\cos x}\,dx$

Let $t = \tan\frac{x}{2}$

$dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx$

$\frac{2\,dt}{1+t^2} = dx$



when $x=0$, $t=0$

$x=\frac{\pi}{2}$, $t=1$

$\cos x = 2\left(\frac{1}{1+t^2}\right) - 1$

$= \frac{1-t^2}{1+t^2}$

$= \int_0^1 \frac{4}{3+5\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2\,dt}{1+t^2}$ (1)

$= \int_0^1 \frac{8\,dt}{3+3t^2+5-5t^2}$ (1)

$= \int_0^1 \frac{8\,dt}{8-2t^2}$

$= \int_0^1 \frac{4\,dt}{(2+t)(2-t)}$ (1)

$\frac{4}{(2+t)(2-t)} = \frac{a}{2+t} + \frac{b}{2-t}$

$= \frac{1}{2+t} + \frac{1}{2-t}$

$= \int_0^1 \left(\frac{1}{2+t} + \frac{1}{2-t}\right) dt$ (1)

$= \ln\left(\frac{2+t}{2-t}\right)\Big]_0^1$

$= \ln 3 - \ln 1$ (1)

$= \ln 3$

c) (i) $I_n = \int_0^3 \frac{x^n}{\sqrt{4-x}}\,dx$

$= \int_0^3 x^n \cdot \frac{d}{dx}(4-x)^{\frac{1}{2}}(-2)\,dx$

($\frac{1}{2}$) $= -2x^n\sqrt{4-x}\Big]_0^3 + 2\int_0^3 \sqrt{4-x}\cdot nx^{n-1}\,dx$

$= -2(3^n - 0) + 2n\int_0^3 \frac{4-x}{\sqrt{4-x}}\,x^{n-1}\,dx$

(1) $= -2(3^n) + 8n\int_0^3 \frac{x^{n-1}}{\sqrt{4-x}}\,dx - 2n\int_0^3 \frac{x^n}{\sqrt{4-x}}\,dx$

$I_n = -2(3^n) + 8n I_{n-1} - 2n I_n$

$I_n(1+2n) = +2\left[4n I_{n-1} - 3^n\right]$

($\frac{1}{2}$) $I_n = \frac{2}{2n+1}\left(4n I_{n-1} - 3^n\right)$

(ii) $I_0 = \int_0^3 \frac{1}{\sqrt{4-x}}\,dx$

$= -2(4-x)^{\frac{1}{2}}\Big]_0^3$

$= 2$ (1)

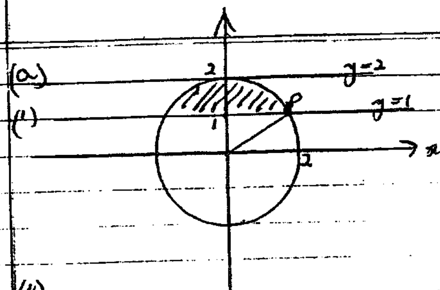
so $I_1 = \frac{2}{2+1}\left(4\times I_0 - 3\right)$

$= \frac{2}{3}(8-3) = \frac{10}{3}$

∴ $I_2 = \frac{2}{5}\left(8\times\frac{10}{3} - 9\right)$ (1)

$= \frac{106}{15}$

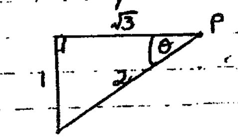
Q2. (a) (i)



$|z| \le 2$ (1)
inside circle

$1 \le \text{Im}\, z \le 2$ (1)
between parallel lines

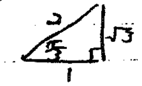
(ii) P is position of number with least argument.



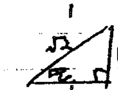
∴ Number is $(\sqrt{3} + i)$ (1)

(b) (i) $z = (1+\sqrt{3}i)(1+i)$
$= (1-\sqrt{3}) + (\sqrt{3}+1)i$ (1)

(ii) $1+\sqrt{3}i = 2\,\text{cis}\,\frac{\pi}{3}$



$1+i = \sqrt{2}\,\text{cis}\,\frac{\pi}{4}$



∴ $z = \left(2\,\text{cis}\,\frac{\pi}{3}\right)\left(\sqrt{2}\,\text{cis}\,\frac{\pi}{4}\right)$ (1)

$= 2\sqrt{2}\,\text{cis}\left(\frac{\pi}{3}+\frac{\pi}{4}\right)$

$= \sqrt{8}\,\text{cis}\,\frac{7\pi}{12}$ (1)

(iii) $z = \sqrt{8}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$

Equating parts

$\cos\frac{7\pi}{12} = \frac{1-\sqrt{3}}{\sqrt{8}}$ (1) & $\sin\frac{7\pi}{12} = \frac{1+\sqrt{3}}{\sqrt{8}}$ (1)

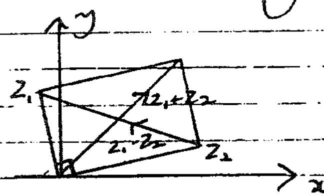
(iv) For z^n to be real, arg is 0 or π
so least n is $\underline{\underline{12}}$ (1)

(c) $z_1 = k i z_2$

Parallelogram of vectors, with z_1 & z_2 at ends of adjacent sides

$\arg\left(\frac{z_1}{z_2}\right) = \pm\frac{\pi}{2}$ $\therefore$ sides perpendicular

(1) $\therefore$ figure is a rectangle.



(z_1+z_2) and (z_1-z_2) represent diagonals, which are (1) equal length

$\therefore |z_1+z_2| = |z_1-z_2|$

(d) (i) $(a+ib)^2 = 5-12i$ where $a, b \in R$

$a^2-b^2=5$ $\quad ab=-6$

$b = -\frac{6}{a}$

$a^2 - \frac{36}{a^2} = 5$

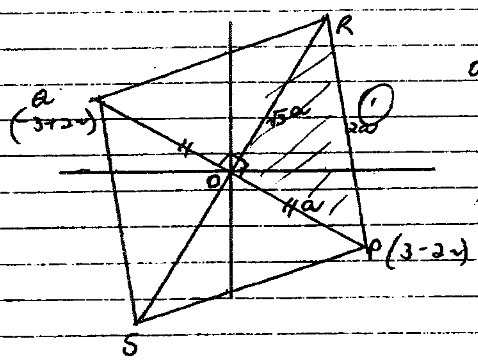
$a^4 - 5a^2 - 36 = 0$ (1)

$(a^2-9)\underbrace{(a^2+4)}_{\text{no sol'n}} = 0$

$\therefore \left.\begin{matrix} a=3 \\ b=-2 \end{matrix}\right\}$ or $\left.\begin{matrix} a=-3 \\ b=2 \end{matrix}\right\}$ (1)

so roots are $3-2i$ or $-3+2i$

(ii)



$\vec{OR} = \vec{OP} * \sqrt{3}\,i$

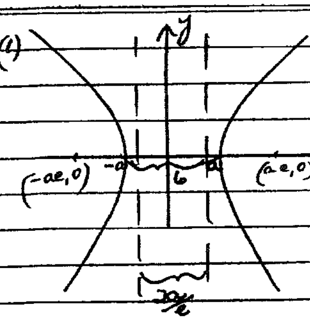
$= (3-2i)(\sqrt{3}\,i)$

$= 2\sqrt{3} + 3\sqrt{3}\,i$

(1) $\therefore R \equiv 2\sqrt{3}+3\sqrt{3}\,i$

$S \equiv -2\sqrt{3} - 3\sqrt{3}\,i$

Q3. (a) (i)



$2a = 6$

$a = 3$

$2ae = 10$ (1)

$\therefore e = 5/3$

$\therefore 2\left(\frac{a}{e}\right) = 2\times3\times\frac{3}{5}$

$= \frac{18}{5}$ (1)

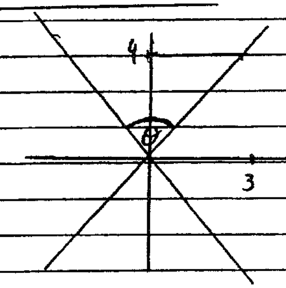
(ii) $e^2 = 1 + \frac{b^2}{a^2}$

$b^2 = a^2(e^2-1)$

$= 9\left(\frac{25}{9}-1\right)$

$= 16$ (1)

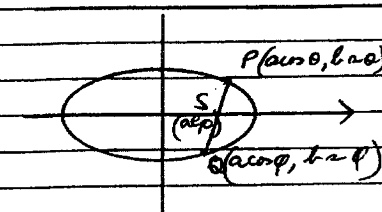
$b = 4$



$\therefore \theta = 2\tan^{-1}\left(\frac{3}{4}\right)$

$= 73^\circ 44'$ (1)

(b)



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$m_{PS} = m_{PQ}$

$\frac{b\sin\theta}{a\cos\theta - ae} = \frac{b(\sin\theta - \sin\varphi)}{a(\cos\theta - \cos\varphi)}$ (1)

$\frac{\sin\theta}{\cos\theta - e} = \frac{\sin\theta - \sin\varphi}{\cos\theta - \cos\varphi}$

$\cos\theta\sin\theta - \cos\theta\sin\varphi - e(\sin\theta - \sin\varphi) = \sin\theta\cos\theta - \sin\theta\cos$

(1) $e = \frac{\sin\theta\cancel{\cos\theta} - \cos\theta\sin\varphi - \sin\theta\cancel{\cos\theta} + \sin\theta\cos}{\sin\theta - \sin\varphi}$

$= \frac{\sin\theta\cos\varphi - \cos\theta\sin\varphi}{\sin\theta - \sin\varphi}$ (1) $\frac{\sin(\theta-\varphi)}{\sin\theta - \sin\varphi}$

(c) $x^3+2x-1=0$

Roots α, β, γ

(i) Roots $-\alpha, -\beta, -\gamma$

$x \Rightarrow -x$

$\therefore (-x)^3+2(-x)-1=0$

$x^3+2x+\ \ =0$ (1)

(ii) Roots $\alpha, -\alpha, \beta, -\beta, \gamma, -\gamma$

$(x^3+2x-1)(x^3+2x+1)=0$ (1)

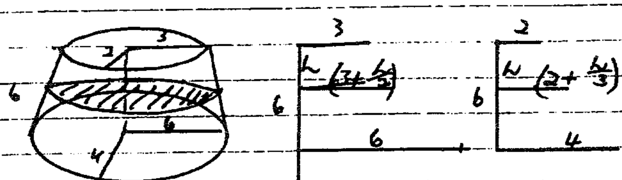
(iii) $\Sigma\alpha\beta \Rightarrow$ want $\frac{c}{a}$ from expansion

or coeff. of x^4: (1)

By inspection, required term is $4x^4$

$\therefore \Sigma\alpha\beta = 4$ (1)

(d)



$\therefore A_{\text{cross section}} = \pi(3+\frac{h}{3})(2+\frac{h}{3})$ (1)

$\therefore \quad \delta V = \pi\left(6+2h+\frac{h^2}{6}\right)\delta h$ (1)

$V = \lim_{\delta h \to 0} \sum_{0}^{6} \pi\left(6+2h+\frac{h^2}{6}\right)\delta h$

$= \pi\int_0^6 \left(6+2h+\frac{h^2}{6}\right)dh$ (1)

$= \pi\left[6h+h^2+\frac{h^3}{18}\right]_0^6$ (1)

$= \pi[36+36+12]$

$= 84\pi$ cub. units.

Q4 (a) $P(x) = x^3+ax^2+bx-10$

$P(3-i)=0 \therefore P(3+i)=0$ (real coeffs)

Zeros α, β, γ

$\alpha\beta\gamma = 10 \Rightarrow (3-i)(3+i)\gamma = 10$ (1)

$10\gamma = 10$

$\gamma = 1$

$\therefore$ Zeros $(3-i), (3+i), 1$

$\Sigma\alpha = -a \Rightarrow (3-i)+(3+i)+1=-a$ (1)

$a=-7$

$\Sigma\alpha\beta = b \Rightarrow (3-i)+(3+i)+10 = b$ (1)

$b = 16$

OR $P(3-i)=0 \therefore P(3+i)=0$

$\therefore (x^2-6x+10)$ is a factor (1)

$\therefore P(x) = x^3+ax^2+bx-10$

$= (x^2-6x+10)(x-1)$ (1)

$= x^3-7x^2+16x-10$

so zeros $(3-i), (3+i)$ and 1

$\therefore a=-7, \ b=16$ (1)

(b)(i) $P(x) = x^3-9x^2+24x+c$

$P'(x) = 3x^2-18x+24$

$= 3(x^2-6x+8) = 0$

when $(x-4)(x-2) = 0$ (1)

$x = 2$ or 4

Repeated root if $P'(x) = P(x) = 0$

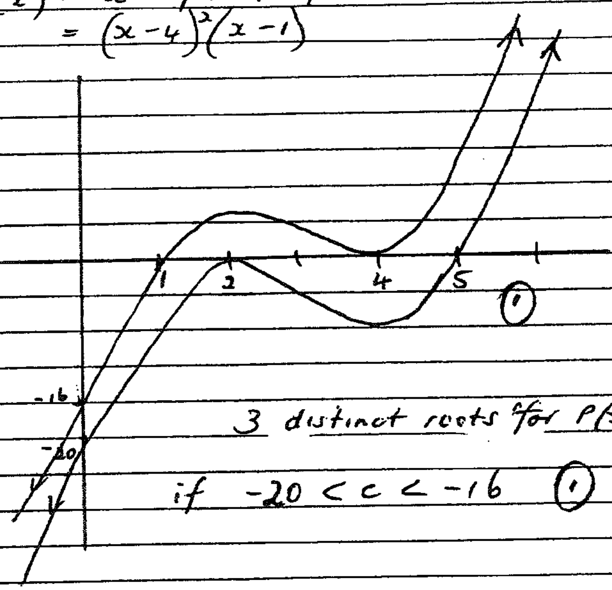
$\therefore P(2)=0 \Rightarrow 8-36+48+c=0$ (1) $\therefore c=-20$

$P(4)=0 \Rightarrow 64-144+96+c=0 \therefore c=-16$

$\therefore c=-16$ or -20.

(ii) $P(x) = x^3 - 9x^2 + 24x - 20$
$= (x-2)^2(x-5)$ (1)

or $P(x) = x^3 - 9x^2 + 24x - 16$
$= (x-4)^2(x-1)$



3 distinct roots for $P(x)=0$

if $-20 < c < -16$ (1)

(c) (i) $y^2 = x^2(x+3)^3$

Domain: $x \geq -3$ since $y^2 \geq 0$

Symmetric about x-axis

x intercepts: $0, -3$

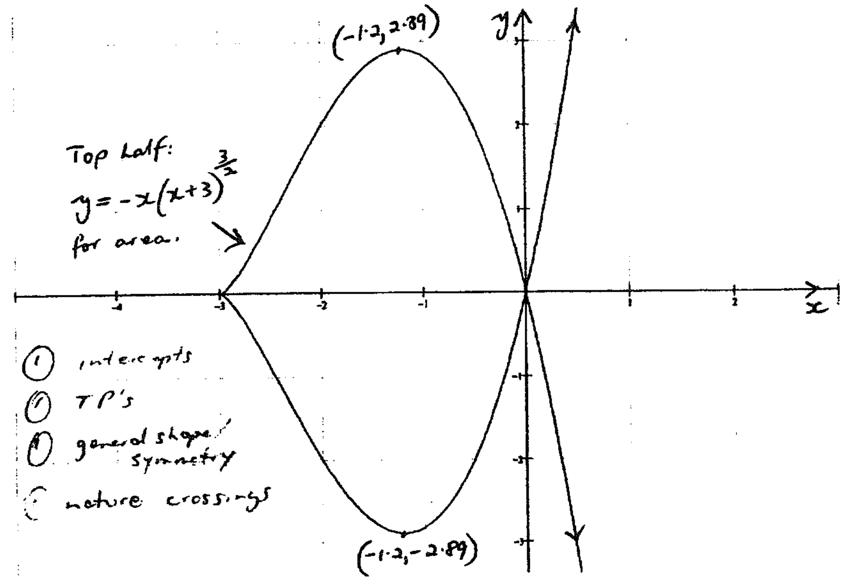
$x \to \infty$, $y \to \infty$ more quickly.

$$2y\frac{dy}{dx} = x^2 \cdot 3(x+3)^2 + 2x(x+3)^3$$
$$= x(x+3)^2[3x + 2(x+3)]$$
$$= x(x+3)^2(5x+6)$$

$\frac{dy}{dx} = 0$ when $x = -\frac{6}{5}$

undefined when $x = 0$ or -3

when $x = -\frac{6}{5}$, $y = 2.89\cdots$ or $-2.89\ldots$

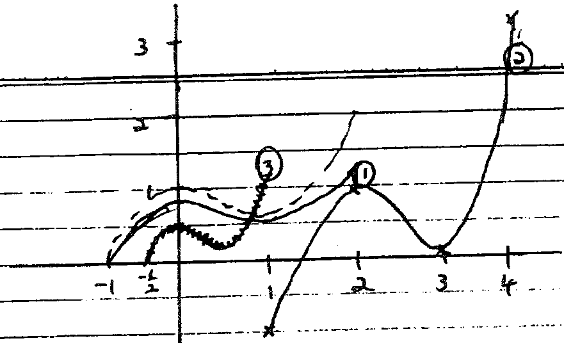


$y^2 = x^2(x+3)^3$

(ii) $A = 2\int_{-3}^{0} -x(x+3)^{\frac{3}{2}}\,dx$ Let $u = x+3$
when $x=0$, $u=3$
$x=-3$, $u=0$

$$= 2\int_0^3 (3-u)u^{\frac{3}{2}}\,du \quad (1)$$
$$= 2\int_0^3 \left(3u^{\frac{3}{2}} - u^{\frac{5}{2}}\right)du$$
$$= 2\left[\frac{6u^{\frac{5}{2}}}{5} - \frac{2u^{\frac{7}{2}}}{7}\right]_0^3 \quad (1)$$
$$= 2\left(\frac{42\times 3^{\frac{5}{2}} - 10\times 3^{\frac{7}{2}}}{35}\right)$$
$$= \frac{2\times 3^{\frac{5}{2}}}{35}(42-30) \quad (1)$$
$$= \frac{216\sqrt{3}}{35}$$

Q5. (a)



(1) $y = \tan^{-1} f(x)$

x	-1	0	2
$f(x)$	0	1	2
$\tan^{-1} f(x)$	0	$\frac{\pi}{4}$ ≈ 0.8	1.1

(1)

(2) $y + 1 = 2f(x-2)$

x	1	2	3	4
$2f(x-2)$	0	2	~ 1	4
y	-1	1	0	3

(1) domain
(1) range

(3) $2y = f(2x)$

x	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
$f(2x)$	0	1	$\sim\frac{1}{2}$	2
y	0	$\frac{1}{2}$	$\sim\frac{1}{4}$	1

(1) domain
(1) range

(b) (i) $\tan^{-1}(n+1) - \tan^{-1} n = \alpha - \beta$ say

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$
$$= \frac{(n+1) - n}{1 + n(n+1)}$$
$$= \frac{1}{n^2 + n + 1}$$

$\therefore \cot(\alpha - \beta) = n^2 + n + 1$ (1)

$\therefore \tan^{-1}(n+1) - \tan^{-1} n = \cot^{-1}(1 + n + n^2)$

(ii) $\cot^{-1}(3) = \tan^{-1} 2 - \tan^{-1} 1$ (since $n = 1$)
$\cot^{-1} 7 = \tan^{-1} 3 - \tan^{-1} 2$ (since $n = 2$)

$\cot^{-1}(1 + n + n^2) = \tan^{-1}(n+1) - \tan^{-1} n$

Adding these:
$\cot^{-1} 3 + \cot^{-1} 7 + \ldots + \cot^{-1}(1 + n + n^2)$
$= (\tan^{-1} 2 - \tan^{-1} 1) + (\tan^{-1} 3 - \tan^{-1} 2) + \ldots + (\tan^{-1}(n+1) - \tan^{-1} n)$
$= \tan^{-1}(n+1) - \tan^{-1} 1$ (1)
$= \tan^{-1}\left[\frac{n+1-1}{1+(n+1)}\right]$
$= \tan^{-1}\left[\frac{n}{n+2}\right]$
$= \cot^{-1}\left[\frac{n+2}{n}\right]$ (1)
$= \cot^{-1}\left(1 + \frac{2}{n}\right)$ as required.

(c) $\frac{x^2}{9} + \frac{y^2}{5} = 1$

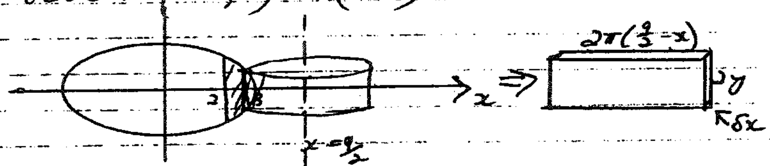
$a = 3, \; b = \sqrt{5}$ $\qquad e^2 = 1 - \frac{b^2}{a^2}$

$= 1 - \frac{5}{9}$

$\therefore \; e = \frac{2}{3}$ (1)

Directrix: $x = 3/\tfrac{2}{3} = \frac{9}{2}$ (1)

Focus: $(ae, 0) \equiv (2, 0)$



A typical shell approximates the prism shown:

$\delta V = 2\pi\left(\frac{9}{2} - x\right).2y\,\delta x$ (1)

$= 4\pi\left(\frac{9}{2} - x\right)\sqrt{5\left(1 - \frac{x^2}{9}\right)}.\;\delta x$

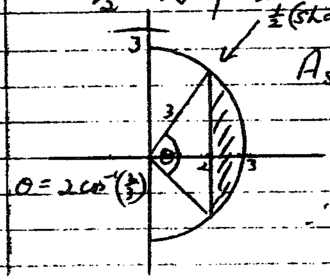
$= \frac{4\sqrt{5}\,\pi}{3}\left(\frac{9}{2} - x\right)\sqrt{9 - x^2}\;\delta x$ (1)

$V = \lim_{\delta x \to 0} \sum_{2}^{3} \delta V$

$= \frac{4\sqrt{5}\,\pi}{3}\left[\frac{9}{2}\int_2^3 \sqrt{9 - x^2}\,dx - \int_2^3 x\sqrt{9 - x^2}\,dx\right]$

$\Downarrow$ I_1 (1) $\qquad$ (1) $\Downarrow$ I_2

$I_1 = \int_2^3 \sqrt{9 - x^2}\,dx$ $\frac{1}{2}$(shaded area)



$A_{seg} = \frac{1}{2}r^2(\theta - \sin\theta)$

$= \frac{1}{2}\cdot 9(1.6821\ldots - 0.9938\ldots)$

$= 3.0973\ldots$

$\therefore \; I_1 = 1.5486\ldots$

$I_2 = \int_2^3 x\sqrt{9 - x^2}\,dx$

$= -\frac{1}{3}\left(9 - x^2\right)^{\frac{3}{2}}\Big]_2^3$

$= -\frac{1}{3}\left(0 - 5\sqrt{5}\right) = \frac{5\sqrt{5}}{3}$

So $V = 6\sqrt{5}\,\pi I_1 - \frac{4\sqrt{5}\,\pi}{3} I_2$

$= 65.27\ldots - 34.906\ldots$

≈ 30.36 (1)

Q6 (a)(i) $z = \cos\theta + i\sin\theta$

$z^n = \cos n\theta + i\sin n\theta$

$z^{-n} = \cos n\theta - i\sin n\theta$ (1)

$z^n + z^{-n} = 2\cos n\theta$

(ii) $z^2 + z^{-2} = 2\cos 2\theta$

$z + z^{-1} = 2\cos\theta$

$5z^4 - 11z^3 + 16z^2 - 11z + 5 = 0$

$\div z^2$: $5z^2 - 11z + 16 - 11\cdot\frac{1}{z} + 5\cdot\frac{1}{z^2} = 0$

$5\left(z^2 + \frac{1}{z^2}\right) - 11\left(z + \frac{1}{z}\right) + 16 = 0$ (1)

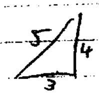
$5(\cancel{2}\cos 2\theta) - 11(\cancel{2}\cos\theta) + \cancel{16}\,8 = 0$

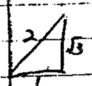
$5(2\cos^2\theta - 1) - 11\cos\theta + 8 = 0$

$10\cos^2\theta - 11\cos\theta + 3 = 0$

$(2\cos\theta - 1)(5\cos\theta - 3) = 0$

$\cos\theta = \frac{1}{2}$ or $\cos\theta = \frac{3}{5}$ (1)

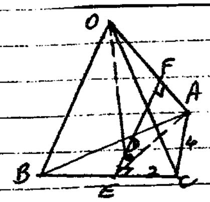




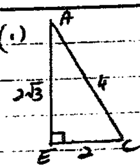
$\sin\theta = \pm\frac{\sqrt{3}}{2}$ or $\sin\theta = \pm\frac{4}{5}$

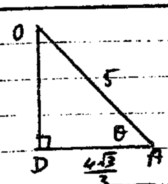
So $z = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ or $\frac{3}{5} \pm i\frac{4}{5}$ (1)

b)



(i)





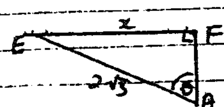
$DA = \frac{2}{3}(AE)$

$= \frac{4\sqrt{3}}{3}$ (1)

$\cos\theta = \frac{4\sqrt{3}}{3\cdot 5} = 0.461\ldots$

$\theta = 62°29'$ (1)

(ii)



(1)

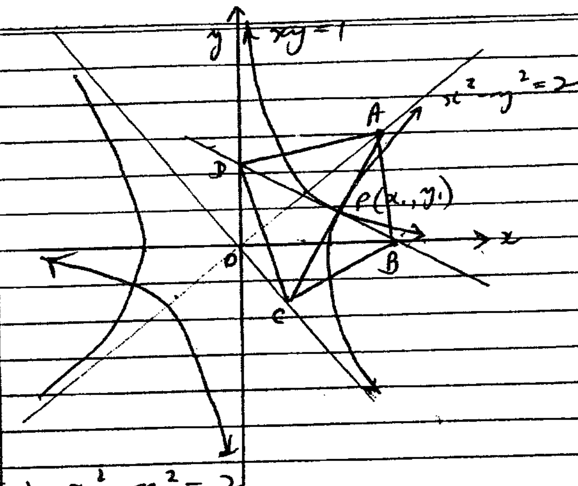
$\sin\theta = \frac{x}{2\sqrt{3}}$

$x = 2\sqrt{3} \times \sin 62°29'$

$= 3.07$ (1)

i.e. dist. from BC to OA is EF i.e. 3.07

(c)



(i) $x^2 - y^2 = 2$

$2x - 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{x}{y} = \frac{x_1}{y_1}$ at P

$\therefore AC$ is $\frac{y - y_1}{x - x_1} = \frac{x_1}{y_1}$

$xx_1 - x_1^2 = yy_1 - y_1^2$

$xx_1 - yy_1 = x_1^2 - y_1^2$

$= 2$ (1)

(ii) $xy = 1 \Rightarrow y = \frac{1}{x}$

$\frac{dy}{dx} = -\frac{1}{x^2} = -\frac{1}{x_1^2}$ at P

so BD is $\frac{y - y_1}{x - x_1} = \frac{-1}{x_1^2}$ $\quad x_1 y_1 = 1$

$(y - y_1)x_1^2 = x_1 - x$ (1)

at B, $y = 0$, $x = x_1 + x_1^2 y_1$

$= x_1 + x_1$ $\quad\therefore B = (2x_1, 0)$

at D, $x = 0$, $y = \frac{1}{x_1} + y_1$ (1)

$= 2y_1$ $\quad\therefore D = (0, 2y_1)$

(ii) at A: $xx_1 - yy_1 = 2$ meets $y = x$

so $xx_1 - xy_1 = 2$

$x(x_1 - y_1) = 2$

$x = \frac{2}{x_1 - y_1} = y$

at C: $xx_1 - yy_1 = 2$ meets $y = -x$

so $xx_1 + xy_1 = 2$

$x(x_1 + y_1) = 2$

$x = \frac{2}{x_1 + y_1}, \quad y = \frac{-2}{x_1 + y_1}$

$\boxed{\text{For a square: diags perp \& bisect each other}}$
& 2 sides ⊥'ar

(1) diags ⊥'ar: $m_{AC} = \dfrac{\frac{2}{x_1 - y_1} + \frac{2}{x_1 + y_1}}{\frac{2}{x_1 - y_1} - \frac{2}{x_1 + y_1}} = \frac{4x_1}{4y_1} = \frac{x_1}{y_1}$

(1) $m_{BD} = \frac{2y_1}{-2x_1} = \frac{-y_1}{x_1}$

$\therefore m_{AC} \times m_{BD} = \frac{x_1}{y_1} \times \frac{-y_1}{x_1} = -1 \quad \therefore$ perp.

(2) Diags bisect: Midpoint $AC = \left\{ \frac{1}{2}\left[\frac{2}{x_1 - y_1} + \frac{2}{x_1 + y_1}\right], \frac{1}{2}\left[\frac{2}{x_1 - y_1} - \frac{2}{x_1 + y_1}\right] \right\}$

(1) $= \left\{ \frac{1}{2} \cdot \frac{4x_1}{x_1^2 - y_1^2}, \frac{1}{2} \cdot \frac{4y_1}{x_1^2 - y_1^2} \right\}$

$= \left(\frac{2x_1}{2}, \frac{2y_1}{2} \right) = (x_1, y_1)$

Midpoint $BD = (x_1, y_1)$ $\therefore$ bisect each other

(3) $m_{AD} \times m_{BC} = \left[\dfrac{\frac{\cancel{2}}{x_1 - y_1} - \cancel{2}y_1}{\frac{\cancel{2}}{x_1 - y_1} - 0} \right] \times \left[\dfrac{\cancel{2}y_1 + \frac{\cancel{2}}{x_1 + y_1}}{0 - \frac{\cancel{2}}{x_1 + y_1}} \right]$

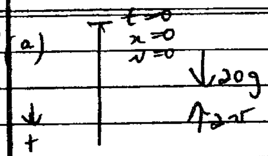
(1) $= \frac{1 - (x_1 y_1 - y_1^2)}{1} \times \frac{x_1 y_1 + y_1^2 + 1}{-1}$

$= y_1^2 \times \frac{2 + y_1^2}{-1}$

$= y_1^2 \times \frac{x_1^2}{-1} = \frac{(x_1 y_1)^2}{-1} = -1$

So ABCD is a square.

Q7 (a)



$20\ddot{x} = 20g - 2v$

$\ddot{x} = 10 - \frac{v}{10}$ (1)

(i) $v_T \Rightarrow \ddot{x} = 0 \quad \therefore v = 100$

(ii) $\frac{dv}{dt} = \frac{100 - v}{10}$

$\frac{dv}{100 - v} = \frac{dt}{10}$

$-\ln(100 - v) = \frac{t}{10} + c$ (1)

when $t = 0, v = 0 \quad \therefore c = -\ln 100$

$\frac{t}{10} = \ln\left(\frac{100}{100 - v}\right)$ (1)

$\frac{100 - v}{100} = e^{-\frac{t}{10}}$

$\frac{v}{100} = 1 - e^{-\frac{t}{10}}$

$v = 100\left(1 - e^{-\frac{t}{10}}\right)$ (1)

(iii) $v \frac{dv}{dx} = \frac{100 - v}{10}$

$\frac{v\,dv}{100 - v} = \frac{dx}{10}$

$\left[\frac{-(100 - v)}{100 - v} + \frac{100}{100 - v} \right] dv = \frac{dx}{10}$ (1)

$-v - 100\ln(100 - v) = \frac{x}{10} + c$ (1)

when $x = 0, v = 0 \quad \therefore c = -100 \ln 100$

$\frac{x}{10} = 100 \ln\left(\frac{100}{100 - v}\right) - v$ (1)

$x = 1000 \ln\left(\frac{100}{100 - v}\right) - 10v$

(iv) when $t=10$, $v = 100(1-e^{-1})$ (1)

so $x = 1000 \ln\left(\frac{100}{100-100(1-e^{-1})}\right) - 10 \cdot 100(1-e$

$= 1000 \ln e - 1000\left(1-\frac{1}{e}\right)$ (1)

$= \frac{1000}{e}$

(b)(i) $\frac{d}{dx}(x^n) = nx^{n-1}$

$\frac{d^2}{dx^2}(x^n) = n(n-1)x^{n-2}$

$\frac{d^3}{dx^3}(x^n) = n(n-1)(n-2)x^{n-3} \ldots$

so $\frac{d^n}{dx^n}(x^n) = n(n-1)(n-2)\cdots(n-n+1)$

$= n!$ (1)

(ii) $\frac{d^n}{dx^n}(x^n \ln x) = n!\left(\ln x + 1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$

Test $n=1$: $\frac{d}{dx}(x \ln x) = x \cdot \frac{1}{x} + \ln x$ (1)

$= 1 + \ln x$

$= 1!(\ln x + 1)$ as required

Let k be a value of n such that

$\frac{d^k}{dx^k}(x^k \ln x) = k!\left(\ln x + 1 + \frac{1}{2} + \cdots + \frac{1}{k}\right)$

then $\frac{d^{k+1}}{dx^{k+1}}(x^{k+1} \ln x) = \frac{d^k}{dx^k}\left(\frac{d}{dx} x^{k+1} \ln x\right)$ (1

$= \frac{d^k}{dx^k}\left[x^{k+1} \cdot \frac{1}{x} + \ln x\,(k+1)\right.$

(1) $= \frac{d^k}{dx^k}\left[x^k + (k+1)\ln x\,(x\right.$

(1) $= k! + (k+1)\left[k!(\ln x + 1 +\right.$

$= (k+1)!\left[\frac{1}{k+1} + \ln x + 1 + \cdots + \frac{1}{k}\right.$

(1) $= (k+1)!\left[\ln x + 1 + \cdots + \frac{1}{k} + \frac{1}{k+1}\right.$

Since this is of required form, propositi[on] is true for $n=k+1$ whenever it is true for $n=k$

Since true for $n=1$, must be true for $n=2$, then true for $n=3$ & so on for all positive integers.

Q8. (a) $\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}$

$\geq 2\sqrt{\frac{a}{b} \cdot \frac{b}{c}} + 2\sqrt{\frac{c}{d} \cdot \frac{d}{a}}$ (1)

$= 2\left[\sqrt{\frac{a}{c}} + \sqrt{\frac{c}{a}}\right]$ (1)

$\geq 2\left[2\sqrt{\frac{a}{c} \cdot \frac{c}{a}}\right]$ (1)

$= 4\sqrt{1}$

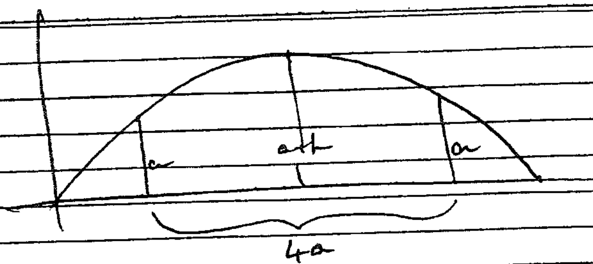
so $\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \geq 4$

AM $\geq$ GM

$\frac{a+b}{2} \geq \sqrt{ab}$

$a+b \geq 2\sqrt{ab}$

(b)



(i) $\dot y = -gt + V\sin\alpha = 0$ when $t = \frac{V\sin\alpha}{g}$

then $y = -\frac{1}{2}gt^2 + Vt\sin\alpha$

$= -\frac{1}{2}g\left(\frac{V^2\sin^2\alpha}{g^2}\right) + \frac{V\sin\alpha \cdot V\sin\alpha}{g}$

$= \frac{-V^2\sin^2\alpha}{2g} + \frac{V^2\sin^2\alpha}{g}$ (1)

so $(a+h) = \frac{V^2\sin^2\alpha}{2g}$

(ii) when $y = a$, $a = -\frac{1}{2}gt^2 + Vt\sin\alpha$

$gt^2 - (2V\sin\alpha)t + 2a = 0$ (1)

$t = \frac{2V\sin\alpha \pm \sqrt{4V^2\sin^2\alpha - 8ag}}{2g}$

so if roots are t_1 and t_2

$t_1 - t_2 = \frac{2\sqrt{4V^2\sin^2\alpha - 8ag}}{2g}$

$= \frac{\cancel{2}\sqrt{4\cdot 2g(a+h) - 8ag}}{\cancel{2}g}$ from (i)

$= \frac{\sqrt{8gh}}{g}$

so $4a = V(t_1 - t_2)\cos\alpha$

$= V\cos\alpha \cdot \sqrt{\frac{8h}{g}}$ (1)

so $V\cos\alpha = \frac{4a\sqrt{g}}{\sqrt{8h}} = a\sqrt{\frac{2g}{h}}$

so $V^2\sin^2\alpha = 2g(a+h)$

$V\cos\alpha = a\sqrt{\frac{2g}{h}} \Rightarrow V^2\cos^2\alpha = a^2\left(\frac{2g}{h}\right)$

$\therefore V^2(\sin^2\alpha + \cos^2\alpha) = 2ga + 2gh + \frac{2ga^2}{h}$

$= 2g\left(a + h + \frac{a^2}{h}\right)$

(1) $V^2 = \frac{2g}{h}\left(h^2 + ah + a^2\right)$

(iii) For V minimum,

$\frac{dV^2}{dh} = 2g\left(1 - \frac{a^2}{h^2}\right)$

$= 0$ when $a = h$ (1)

$\frac{d^2V^2}{dh^2} = -2ga^2 \cdot -2h^{-3}$

$= \frac{4ga^2}{h^3} > 0$

$\therefore$ min $V^2 \Rightarrow$ min V when $a = h$ (1)

then $t_2 = \frac{2V\sin\alpha - \sqrt{4V^2\sin^2\alpha - 8ag}}{2g}$

so $x = Vt_2\cos\alpha$

$= V\cos\alpha \cdot t_2$

$= a\sqrt{\frac{2g}{a}} \cdot \frac{2\sqrt{4ag} - \sqrt{16ag - 8ag}}{2g}$ (1)

$= a\sqrt{\frac{2g}{a}} \cdot \frac{\cancel{2}\cdot 2\sqrt{ag} - \cancel{2}\sqrt{2ag}}{\cancel{2}g}$

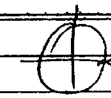
$= \sqrt{2a\cancel{g}} \cdot \frac{\sqrt{a\cancel{g}}(2 - \sqrt{2})}{\cancel{g}}$

$= \sqrt{2}\,a(2 - \sqrt{2})$ (1)

$= (2\sqrt{2} - 2)a$

$= 2(\sqrt{2} - 1)a$

(c)(i) $z = \cos\theta + i\sin\theta$



$z = 1 \Rightarrow \theta = 0$ (1)

(ii) $\dfrac{dz}{d\theta} = -\sin\theta + i\cos\theta$

$= i(\cos\theta + i\sin\theta)$ (1)

$= iz$

(iii) $\dfrac{d\theta}{dz} = \dfrac{1}{i}\cdot\dfrac{1}{z}$

$\theta = \frac{1}{i}\ln z + c$

when $\theta = 0,\ z = 1 \Rightarrow c = 0$

$\theta = \frac{1}{i}\ln z$ (1)

$i\theta = \ln z$

$e^{i\theta} = z \;(= \cos\theta + i\sin\theta)$

$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$

$= \cos\theta - i\sin\theta$ (1)

$\therefore 2\cos\theta = e^{i\theta} + e^{-i\theta}$

$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$